Period-doubled breathing in trapped Bose-Einstein condensates

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The response of a trapped Bose-Einstein condensed gas to a periodic driving force is studied theoretically in the framework of the nonlinear Gross-Pitaevskii equation. The monopole mode is driven by periodical modulation of the frequency of the isotropic harmonic trapping potential. Period-doubled motion is shown to occur when the driving is strong and close to resonance with the monopole mode. A variational model predicts chaotic oscillations but is found to disagree with the full solutions to the Gross-Pitaevskii equation.

A trapped Bose-Einstein condensed gas is a damped, nonlinear oscillator. The modes of oscillation result from an interplay between dispersion, the repulsive inter-particle interactions and the confining trap potential [1], and the damping is offered by friction against thermal atoms [2, 3]. Interesting physics can therefore be anticipated if the trap is modulated so as to drive the modes of oscillation: periodically forced, nonlinear oscillators can generally be expected to exhibit period-doubled motion and chaos for certain parameter values. Textbook examples include the Duffing oscillator and the forced, damped pendulum [4].

In this paper, we shall study the oscillations of Bose-Einstein condensed gases subject to periodic driving forces, with particular attention to the conditions for period-doubled motion and the onset of chaos. The basis for all calculations is the Gross-Pitaevskii equation (GPE) for the condensate wave function $\psi(\mathbf{r},t)$, [5, 6]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r},t)\psi + \frac{4\pi\hbar^2 a}{m}|\psi|^2\psi. \quad (1)$$

The wave function $\psi(\mathbf{r},t)$ determines the local fluid density and velocity through $n(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and $\mathbf{v}(\mathbf{r},t) = (\hbar/m)\nabla\arg\psi(\mathbf{r},t)$; it is normalized to the number of condensed atoms N. The nonlinear term in the GPE describes inter-particle interactions in the s-wave approximation and its strength is determined by the s-wave scattering length a. The trap potential $V(\mathbf{r},t)$ is a harmonic-oscillator potential which in this study shall be taken to be isotropic and periodically modulated in time:

$$V(\mathbf{r},t) = \frac{1}{2}m\omega^2 r^2 \left(1 + \Delta\cos\Omega t\right). \quad (2)$$

Assuming that the density profile stays fixed and only its overall radial extent is allowed to vary, the cloud size is described by an effective equation [7],

$$\ddot{R} = -R\left(1 + \Delta\cos\bar{\Omega}\bar{t}\right) + \frac{1}{R^4} - 2\gamma\dot{R}. \quad (3)$$

This equation is often derived in a variational scheme [8] and will therefore be referred to as "variational" in the

following. The first term on the right hand side represents the trap potential including the periodic modulation. The inverse-power term originates from the inter-particle interactions. The kinetic energy associated with the density gradient is assumed negligible in comparison with the trap and interaction energies and the corresponding term is therefore not included in Eq. (3); this approximation holds when the scattering length is large and the mean density is high [8]. The unit of length is taken to be the Thomas-Fermi radius $R_{\mathrm{TF}} = a_{\mathrm{osc}}(15Na/a_{\mathrm{osc}})^{1/5}$, where $a_{\mathrm{osc}} = (\hbar/m\omega)^{1/2}$ is the oscillator length, and the time and frequency have been scaled by the trap frequency ω, so that $\bar{\Omega} = \Omega/\omega$ and $\bar{t} = \omega t$. We shall hereafter work only in these dimensionless units, and we therefore immediately drop the bar on t and Ω. The final linear friction term in Eq. (3) is introduced by hand and represents the dissipation due to the thermal gas. For the physical situation at hand, damping plays a pivotal role and it is therefore important to allow for this effect. This phenomenological treatment of the dissipation is known to well reproduce the experimental findings of Ref. [2]. The friction is provided by Landau damping and depends on temperature through the relation [3]

$$\frac{\gamma}{\omega_m} = 0.645\frac{1}{N^{2/3}}\left(\frac{Na}{a_{\mathrm{osc}}}\right)^{4/5}\frac{T/T_c}{\left[1 - (T/T_c)^3\right]^{1/5}}, \quad (4)$$

where ω_m is the frequency of the mode in question, and T_c is the critical Bose-Einstein condensation temperature. The damping coefficient $\gamma = 0.02$, which is used in much of the calculations of this paper, corresponds to a cloud of $N = 10^6$ atoms with $a_{\mathrm{osc}}/a = 100$ at a temperature $T = 0.1T_c$.

The virtue of the variational equation (3) is of course its simplicity, allowing for fast numerical solution, so that large parameter spaces can be covered, and in a few instances the existence of analytical solutions. However, the results obtained this way must of course be double-checked against the solution of the full Gross-Pitaevskii equation, Eq. (1), to find out whether deformation of the density profile plays a crucial role. For the present case of isotropic forcing, we shall assume that the cloud stays spherically symmetric. The Gross-Pitaevskii equation, thus made effectively one-dimensional, is propagated in time with the Crank-Nicolson method. Dissi-

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pation is represented by letting the time be complex, so that in solving the discretized equation, the time step is $\Delta t = \Delta t_R (1 + i\Gamma_{\rm GP})$. It is found numerically that the parameter $\Gamma_{\rm GP}$ is related to the damping of the breathing mode through $\gamma \approx 3\Gamma_{\rm GP}$.

The problem depends on three physical parameters: the amplitude Δ of the perturbation, the frequency Ω of the perturbation, and the damping γ [9]. Luckily, the dynamics of the cloud is found not to depend sensitively on the latter, which is difficult to control or even measure with great precision. When the system is not driven, $\Delta = 0$, the system performs damped sinusoidal oscillations with the frequency of the breathing mode, which is equal to $\sqrt{5}$ in the strong-coupling limit considered here. When the amplitude of the driving force Δ is nonzero, but not too large, we expect the motion to be frequency locked, i. e. it performs breathing motion that is periodic with the frequency Ω, or possibly fractions thereof, in which case we have period-multiplied motion. When $\Delta > 1$, the instantaneous trapping potential will for a certain time interval during each period not be confining, because the instantaneous squared trap frequency, $\omega^2(1 + \Delta \cos \Omega t)$, turns negative. This does not, however, necessarily imply that atoms are lost, because such events happen on a finite time scale: if the trap is opened but then sufficiently rapidly closed again, the atoms remain confined. For large enough Δ, the system does indeed turn unstable, but the threshold value depends sensitively on Ω (and less sensitively on γ).

The region of instability of the variational Eq. (3) have been examined in Refs. [10, 11]. For small forcing amplitudes Δ, the instabilities occur in the vicinity of the frequency values $\Omega = 2/n$, $n = 1, 2, 3 \ldots$, and as Δ is increased the instability regions grow larger. Since instability is associated with large-amplitude motion, the relevant frequency is in fact the bare trap frequency (which is 1 in the units currently employed), rather than the quadrupole frequency $\sqrt{5}$, and the instability regime is the same as that for the Mathieu equation. The inclusion of damping somewhat shrinks the instability regimes. This analysis predicts an appreciable domain for stability even for values of Δ well above unity, especially when Ω is large. In fact, when Ω approaches infinity, the threshold value of Δ for instability also approaches infinity.

Figure 1 shows examples of the time evolution of $R(t)$, the solution of Eq. (3), for two instances of the values of the parameters Ω and Δ. The limit cycle, i. e. the steady orbit that is approached at long times, in the phase space spanned by R and $\dot{R}$ is also shown. The damping parameter is fixed at $\gamma = 0.02$. In the first example, the forcing frequency $\Omega = 2.2$ is comparable to the quadrupole frequency and the forcing amplitude is rather low, and therefore the motion is in this case nearly sinusoidal (so that the limit cycle takes on a nearly elliptic shape) and locked to the forcing frequency. For a larger amplitude Δ the shape of the oscillations is distorted due to the strongly anharmonic effective potential well that R resides in. When the forcing frequency is
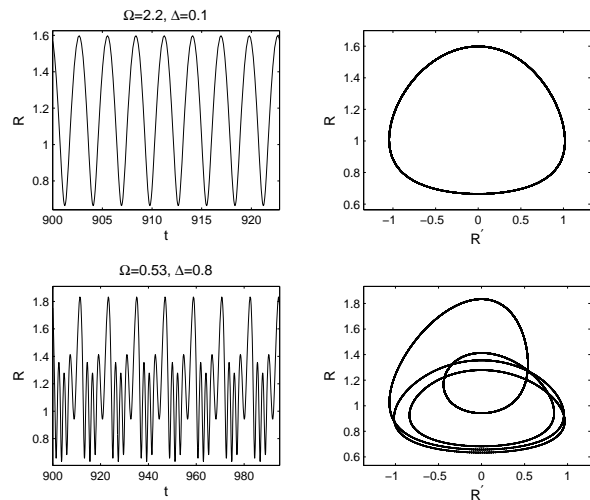

FIG. 1: Time evolution and limit cycles of the forced condensate, with driving amplitude Δ and frequency Ω as indicated above the panels. The leftmost plots display a time span of eight periods of the driving force.

smaller than an integer fraction of the breathing-mode frequency, $\Omega < \sqrt{5}/n$, an n-th harmonic is seen if Δ is large enough (but the motion is still periodic with period Ω). This is in Fig. 1 illustrated for $\Omega = 0.53 = \sqrt{5}/4.2$ where four loops are seen in the limit cycle. These variational results have been compared to the solution of the full GPE (1), and are in excellent agreement.

For larger values of the forcing amplitude Δ more complex behavior is observed. When Ω is close to resonance with the breathing-mode frequency $\Omega = \sqrt{5}$ or fractions thereof [12], and Δ is large enough, the variational system (3) exercises period-doubling bifurcations and chaotic motion. Figure 2 shows the period-2 motion seen at $\Omega = 1.25 \approx \sqrt{5}/2$, $\gamma = 0.02$ and $\Delta = 1.5$. Shown are both the variational result for R and the rms radius, $\sqrt{\langle r^2 \rangle}$, of the cloud as calculated using the Gross-Pitaevskii equation, scaled by the Thomas-Fermi value $R_{rms} = \sqrt{3/7} R_{\rm TF}$ to make the scale agree with that of the variational study. The limit cycle is plotted in the space of $R(t)$ and $R(t - \Delta t)$, where Δt is a small time delay; this yields an image tilted by 45 degrees compared to that in R-$\dot{R}$ space. The variational model still agrees well with the full GPE. However, concerning chaos the situation is different: the solutions to the full GPE fail to exhibit chaotic motion when the variational does. An example is shown in Fig. 3, where the parameters $\Omega = 0.7$, $\Delta = 1.8$ and $\gamma = 0.02$ are chosen. That the solution to the effective-radius equation is indeed chaotic has been checked by calculating the Lyapunov exponent. However, the solution to the full Gross-Pitaevskii equation is seen to be perfectly regular: still period-doubled, albeit with a rather large amplitude. The grid parameters have been chosen so that the wave function both in configuration and momentum space stays well inside the grid boundaries at all times; grids up to 1500 points with a
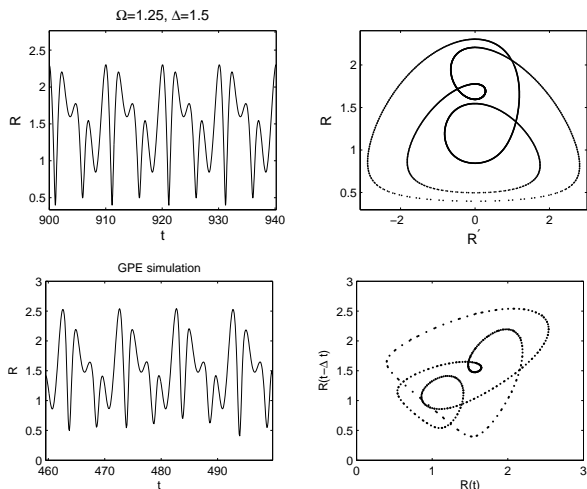
FIG. 2: Time evolution and limit cycles of the forced condensate. Variational results are shown in the upper panels, while the lower panels show the results of the full Gross-Pitaevskii equation.The delay parameter for the limit cycle in the lower plot is chosen as $\Delta t = 0.5$.
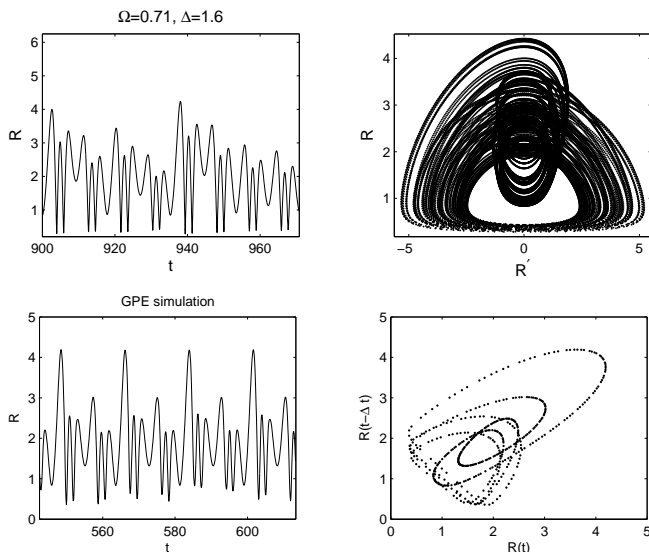

FIG. 3: Time evolution and limit cycles of the forced condensate. Variational results are shown in the upper panels, while the lower panels show the results of the full Gross-Pitaevskii equation.

grid constant $\Delta x = 0.04 a_{\mathrm{osc}}$ have been tried.

The result is quite general: nowhere in phase space does chaotic motion, or even periodic motion with period longer than 2, occur in the GPE solution. The discrepancy is not due to the omission of the dispersion-related kinetic-energy term in Eq. (3); that is readily checked by including such a term whereby the solution still exhibits chaotic motion. Rather, the fallacy of the variational Eq. (3) is that it does not let the wave function change its shape during the dramatic changes of the trap potential
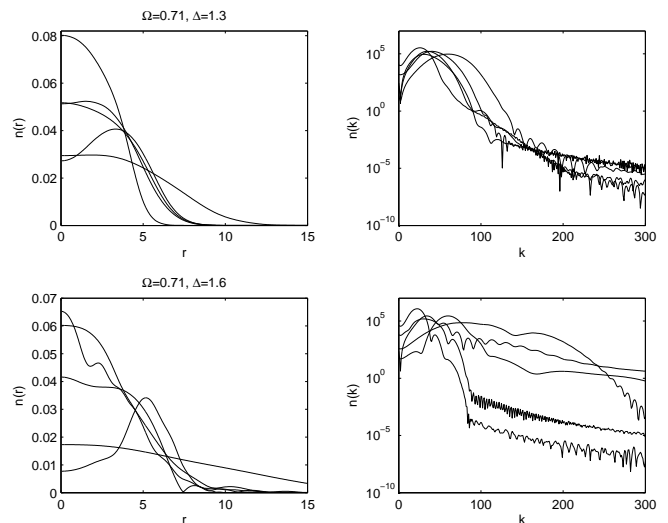

FIG. 4: Density distribution at a few time instances, as calculated by the full Gross-Pitaevskii equation. Radial real-space density distributions are shown to the left and momentum-space distributions to the right. Topmost panels: periodic motion with $\Omega = 0.71$ and $\Delta = 1.3$. Lower panels: period-two motion with $\Omega = 0.71$ and $\Delta = 1.6$; this case was predicted by the variational equation to be chaotic. The different curves in each panel are chosen at arbitrary equally spaced instances, but the same set of time instances were used in all panels.

between confining and repelling. This may come as a surprise, because a variational study is almost always seen to faithfully mimic the physics at least qualitatively [7]. A closer look at the wave function gives an explanation. Figure 4 shows a few snapshots of the wave function for the cases of period-one and period-two motion. In the regime where there is agreement between the variational equation and the GPE, the profile of the wave function does not change drastically, in accordance with the assumptions behind the variational equation, but in the regime predicted to be chaotic, waves propagate back and forth through the cloud. Clearly the variational system has too few degrees of freedom to faithfully model the behavior of the cloud in this regime. In a manner of speaking, the instability that manifests itself in short-wavelength density waves in the full GPE study, must instead find its outlet as chaotic motion in the restricted variational study.

We conclude by showing in Fig. 5 the full phase diagram of the forced system as calculated by the variational equation. The damping parameter is still $\gamma = 0.02$. Small regions of period-multiplied motion and chaos are seen to exist around the stability tongues just above $\Omega = \sqrt{5}/n$ for all integers $n > 2$. The largest chaotic regime is also indicated although it was found to be an artifact of the variational equation.

To summarize, the Gross-Pitaevskii equation in a trap with a periodic modulation of the trapping potential has been seen to support period-doubled oscillations, but chaotic motion was not observed. In contrast, a sim-
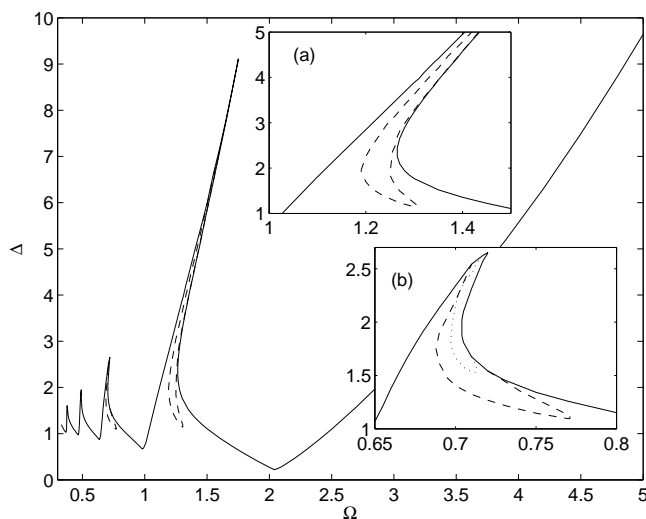
FIG. 5: Phase diagram for the forced condensate. Solid lines enclose the regime of stable motion; dashed lines enclose the regimes of period-doubled motion and the dotted line encloses the regime where the variational study displays chaotic motion. The insets labeled (a) and (b) zoom in on two important regions.

plified description of the same system using a collective coordinate predicts chaotic motion and period-multiplied motion with period higher than two, because of its failure to describe short-wavelength density modulations. Indeed, it was suggested in Refs. [11, 13] that chaotic mo-

tion cannot occur in Bose-Einstein condensed systems; however, it seems that the reasoning leading to that conclusion is not applicable in the present case. Reference [13] proposed that the exponentially growing separation between neighbouring orbits in phase space that is characteristic of chaos implies that the depletion of the condensate grows equally fast, and Ref. [11] suggested that this can be taken as proof that chaotic oscillation in Bose-Einstein condensates is impossible. However, said analysis does not address the case of bounded chaotic oscillations where the exponential separation of nearby orbits only takes place over short times while at long times the whole bundle of orbits stays bounded, as in Fig. 3. Furthermore, if the excited modes are coherent with the condensate, it is not at all clear that their growth must be identified with depletion of the condensate. The absence of chaos observed here should therefore be seen as an accidental fact, and the question of whether chaotic oscillations in fact can be observed for slightly different physical situations – for example exploring other parameter values or driving modes with higher multipolarity – still remains open.

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